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The zeros of linear combinations of orthogonal polynomials

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Abstract

Let $\{p_n\}$ be a sequence of monic polynomials with p_n of degree n, that are orthogonal with respect to a suitable Borel measure on the real line. Stieltjes showed that if $m < n$ and $x_1, \ldots, x_n$ are the zeros of p_n with $x_1 < \cdots < x_n$ then there are m distinct intervals f the form (x_j, x_{j+1}) each containing one zero of p_m. Our main theorem proves a similar result with p_m replaced by some linear combinations of $p_1, \ldots, p_m$. The interlacing of the zeros of linear combinations of two and three adjacent orthogonal polynomials is also discussed.

MSC: primary 33C50; secondary 33D50

Keywords: Orthogonal polynomials; Linear combinations; Interlacing zeros

1. Introduction

This paper examines interlacing properties of the zeros of linear combinations of orthogonal polynomials. Throughout this paper, μ will denote a positive Borel measure on $\mathbb{R}$ with the property that every real polynomial p is μ-integrable over $\mathbb{R}$ with $\int p^2 d\mu > 0$ unless p is the zero polynomial. We shall say that such a measure μ is *admissible*. Any admissible measure μ induces a scalar product $\langle p, q\rangle$ (the integral of the product pq with respect to μ) on the vector space $\mathcal{P}$ of real

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polynomials, and this leads to a unique sequence $p_0, p_1, \dots$ of mutually orthogonal polynomials, where p_n is monic and of degree n. It is well known that the n zeros of p_n are real and distinct, and that they lie in the interior $I(\mu)$ of the convex hull of the support of μ. In his fundamental paper [10] Shohat considered linear combinations of orthogonal polynomials, and he observed ([10], p. 465) that the usual proof that p_n has its n zeros in $I(\mu)$ actually proves the stronger statement that any linear combination $P = a_s p_s + \cdots + a_n p_n$, where $s \leqslant n$ and $a_s a_n \neq 0$, has at least s distinct zeros in $I(\mu)$. This result is best possible as can be seen, for example, from the discussion in Chapter 6, [11], regarding the zeros of the Jacobi and Laguerre polynomials (see also Section 7 of this paper). We note that the existence of at least s odd order zeros of P in $I(\mu)$ is also an immediate consequence of the quasi-orthogonality of P (cf. [2]).

It is well known that the $n-1$ zeros of p_{n-1} are distinct from those of p_n, and they interlace with the zeros of p_n in the sense that exactly one zero of p_{n-1} lies between any two consecutive zeros of p_n. Stieltjes proved a stronger result than this, namely that the zeros of p_m and p_n are interlaced whenever $m < n$ in the sense that if $x_1, \dots, x_n$ are the zeros of p_n with $x_1 < \cdots < x_n$, then there are m disjoint intervals of the form (x_j, x_{j+1}) such that each contains a zero of p_m (see [1], p. 253 and [11], Theorem 3.3.3, p. 45). Recently, Gibson [5] noted that Stieltjes' result implies that p_m and p_{m+k} have at most $\min\{m, k-1\}$ common zeros. The interlacing of zeros of two polynomials of consecutive degrees is useful in other contexts. For example, it occurs in connection with associated polynomials; see [8]. Interlacing also plays a role in the study of polynomials satisfying a three-term recurrence relation (see [8], [12]).

Our main result (Theorem 3.1) extends Stieltjes' result, and is concerned with the interlacing of the zeros of the polynomial ω used to generate a quadrature formula (where ω may, but need not, be p_n) and the zeros of the linear combination $a_s p_s + \cdots + a_m p_m$, where $s \leqslant m \leqslant n$. Theorem 3.1 is stated and proved in Section 3, following a brief discussion in Section 2 of the degree of precision of a quadrature formula. In Section 4, we recall some standard results of the Wronskian of two polynomials, which we use in Sections 5 and 6 to discuss linear combinations of two and three adjacent orthogonal polynomials, respectively. In Section 7, we comment on some results in Szegö [11] and Shohat [10].

2. The degree of precision

Let $\omega(x) = (x - c_1) \cdots (x - c_n)$, where the c_j are any real numbers with $c_1 < \cdots < c_n$. A *quadrature formula* for the triple (ω, μ, λ), where μ is an admissible measure μ, and $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$, is a formula of the form

$$\int_{\mathbb{R}} p \, d\mu = \sum_{j=1}^{n} \lambda_j p(c_j) \tag{2.1}$$

that is valid for all polynomials p in some class $\mathcal{Q}$. Obviously, one wants to identify the class $\mathcal{Q}$ once μ, ω and the λ_j are given. It is clear that (2.1) fails when $p(x) = \omega(x)^2$, which is of degree $2n$, regardless of μ and the λ_j, because the integral in (2.1) is positive while the sum is zero. On the other hand, (2.1) holds for all polynomials p of degree at most $n-1$ providing that we make an appropriate choice of the λ_j. Lagrange's interpolation formula shows that there are unique polynomials π_j, of degree $n-1$, such that $\pi_j(c_j) = 1$ and $\pi_j(c_k) = 0$ when $k \neq j$, and we use these to define the *Cotes numbers* λ_j by

$$\lambda_j = \int_{\mathbb{R}} \pi_j \, d\mu. \tag{2.2}$$

The operator $p \mapsto p^*$, defined by

$$p^*(x) = \sum_{j=1}^{n} \pi_j(x) p(c_j),$$

has the property that $p^*(c_k) = p(c_k)$ for $k = 1, \ldots, n$. It follows immediately that if p is any polynomial of degree at most $n-1$, then $p^* = p$ and (2.1) holds since

$$\int_{\mathbb{R}} p \, d\mu = \int_{\mathbb{R}} p^* \, d\mu = \sum_{j=1}^{n} p(c_j) \int_{\mathbb{R}} \pi_j \, d\mu = \sum_{j=1}^{n} \lambda_j p(c_j).$$

This discussion shows that, given the pair (ω, μ), and the λ_j defined by (2.2), there is an integer $d(\omega, \mu)$ that satisfies

$$n - 1 \leqslant d(\omega, \mu) \leqslant 2n - 1$$

and which is such that (2.1) holds for all polynomials of degree $d(\omega, \mu)$ but not for all polynomials of higher degree. Shohat calls $d(\omega, \mu)$ the *degree of precision* of (ω, μ). Roughly speaking, the degree of precision increases as the relationship between ω and the orthogonal p_n associated with μ increases. In general, $d(\omega, \mu) = n - 1$. In Theorem 1 ([10], p. 465) Shohat proves that (ω, μ) has degree of precision $(n-1) + s$ if and only if $\omega = a_s p_s + \cdots + a_n p_n$, where $a_s a_n \neq 0$. This tells us that the degree of precision is $2n - 1$ if and only if $\omega = p_n$, and we may view this as an algorithm to find the optimal choice for ω for a given μ rather than as an application of the quadrature formula to orthogonal polynomials. Of course, Shohat's result shows that if the degree of precision is $n - 1$ then ω is a linear combination of $p_0, \ldots, p_n$, and hence can be any polynomial of degree n.

3. The main result

Our main result suggests that the property of interlacing of zeros is inherited from the positivity of the Cotes numbers in a quadrature formula rather than directly from orthogonality.

Theorem 3.1. *Let μ be an admissible measure and let $\{p_k\}$ be the sequence of monic μ-orthogonal polynomials. Let*

$$\omega(x) = (x - c_1) \cdots (x - c_n),$$

where $c_1 < \cdots < c_n$, and suppose that all of the Cotes numbers λ_j defined by (2.2) *are positive. Let $P = a_s p_s + \cdots + a_m p_m$, where $a_s a_m \neq 0$, $1 \leqslant s \leqslant m \leqslant n$, and $m \leqslant d(\omega, \mu)$. Then either* (i) *$P$ is a non-zero scalar multiple of ω, or* (ii) *at least N of the intervals (c_j, c_{j+1}) contain a zero of P, where $N = \min\{s, d(\omega, \mu) + 1 - m\} \geqslant 1$.*

Proof. Take any polynomial Q such that $\deg(Q) < N$; then

$$\deg(PQ) = \deg(P) + \deg(Q) \leqslant m + (N - 1) \leqslant d(\omega, \mu),$$

so that the quadrature formula applies to PQ. Because $\deg(Q) < N \leqslant s$ we see that Q is orthogonal to P, and hence

$$\sum_{j=1}^{n} \lambda_j P(c_j) Q(c_j) = \int_{\mathbb{R}} PQ \, d\mu = 0. \tag{3.1}$$

This holds for any Q with $\deg(Q) < N$.

We suppose now that (i) fails, and we show that (ii) holds. As (i) fails and $\deg(P) \leqslant \deg(\omega)$, there is some c_j such that $P(c_j) \neq 0$. Suppose, for the moment, that there is only one value of j such that $P(c_j) \neq 0$, and let this value be ℓ. We now take $Q \equiv 1$. Then, as $\deg(Q) = 0 < 1 \leqslant N$, we see that (3.1) holds and we obtain

$$0 = \sum_{j=1}^{n} \lambda_j P(c_j) = \lambda_\ell P(c_\ell) \neq 0.$$

We conclude that there are at least two zeros c_j of ω that are not zeros of P. Now let the set of zeros of ω that are not zeros of P be $c_{j_1}, \ldots, c_{j_t}$, where $t \geqslant 2$ and $j_1 < j_2 < \cdots < j_t$, and, for brevity, write $X_k = c_{j_k}$. Then, for any Q with $\deg(Q) < N$,

$$\sum_{k=1}^{t} \lambda_{j_k} P(X_k) Q(X_k) = \sum_{j=1}^{n} \lambda_j P(c_j) Q(c_j) = 0. \tag{3.2}$$

Now $X_1 < \cdots < X_t$, where $t \geqslant 2$. For each $k = 1, \ldots, t-1$, let y_k be any point in the interval (X_k, X_{k+1}). For each k, $P(X_k)P(X_{k+1})$ is either positive or negative, and we let $\sigma(x) = \prod(x - y_k)$, where this product is over all k for which $P(X_k)P(X_{k+1}) < 0$. If no such k exist we put $\sigma(x) \equiv 1$. Thus σ has a single (and simple) zero in (X_k, X_{k+1}) if and only if $P(X_k)P(X_{k+1}) < 0$. It follows that for each k, $\sigma(X_k)\sigma(X_{k+1})$ and $P(X_k)P(X_{k+1})$ have the same sign, so that

$$\sigma(X_k)P(X_k)\sigma(X_{k+1})P(X_{k+1}) > 0, \quad k = 1, \ldots, t-1.$$

We deduce that, for each k, the non-zero terms $P(X_k)\sigma(X_k)$ and $P(X_{k+1})\sigma(X_{k+1})$ have the same sign, and this implies that

$$\sum_{k=1}^{t} \lambda_{j_k} P(X_k)\sigma(X_k) \neq 0, \tag{3.3}$$

because we have assumed that the λ_j are positive. This shows that

$$\deg(\sigma) \geqslant N = \min\{s, d(\omega, \mu) + 1 - m\} \tag{3.4}$$

because otherwise, we could take $Q = \sigma$ and obtain a contradiction from (3.2) and (3.3). As (3.4) implies the conclusion of Theorem 3.1, this completes the proof. □

Remark. For each $j = 1, \ldots, n$, the Cotes number λ_j defined by (2.2) depends on μ and the interpolation points (or nodes) $c_1, \ldots, c_n$. A central assumption in Theorem 3.1 is the positivity of λ_j for $j = 1, \ldots, n$. It is well known that if $\omega = p_n$, or if $\omega = p_n + ap_{n-1}$ ([7], Theorem III), then all the Cotes numbers are positive. In [6], Peherstorfer gives a characterization of the real numbers $c_1, \ldots, c_n$ which ensures the positivity of the Cotes numbers λ_j. Further, in [7] with

$I(\mu) = (-1, 1)$, Peherstorfer derives sufficient conditions on the real numbers a_j, $j = 0, \ldots, m$ such that the linear combination of orthogonal polynomials $\hat{P} = a_0 p_n + \cdots + a_m p_{n-m}$ has n simple zeros in $(-1, 1)$ and the Cotes numbers corresponding to Lagrange interpolation at the zeros of $\hat{P}$ are positive.

Theorem 3.1 contains the following generalization of Stieltjes' result (which is the case $1 \leqslant s = m < n$ below).

Theorem 3.2. *Let $\{p_k\}$ be a μ-orthogonal sequence of monic polynomials, with p_k of degree k, and suppose that $x_1, \ldots, x_n$ are the zeros of p_n with $x_1 < \cdots < x_n$. Let $P = a_s p_s + \cdots + a_m p_m$, where $a_s a_m \neq 0$, $1 \leqslant s \leqslant m \leqslant n$ and $s < n$. Then there are at least s disjoint intervals (x_j, x_{j+1}) that contain at least one zero of P.*

Proof. We take $\omega = p_n$ in Theorem 3.1. Then $x_j = c_j$ for each j, and $m \leqslant n \leqslant 2n - 1 = d(\omega, \mu)$. The fact that $s < n$ means that P is not a scalar multiple of ω (which is p_n), and this eliminates the possibility (i) in Theorem 3.1. Finally, as $2n > m + s$, we have

$$d(\omega, \mu) + 1 - m = (2n - 1) + 1 - m > s,$$

so that $N = s$. The conclusion now follows from Theorem 3.1(ii). □

Moreover, it is well known that when $\omega = p_n$ all of the λ_j are positive.

4. The Wronskian

We recall some known facts about the Wronskian

$$W(x; p, q) = \begin{vmatrix} p(x) & q(x) \\ p'(x) & q'(x) \end{vmatrix} = p(x)q'(x) - p'(x)q(x)$$

of two polynomials p and q. Since the Wronskian is linear in p and in q, it is potentially useful in any discussion of linear combinations of polynomials. Also, if $W(x; p, q)$ is non-zero in an open interval J, then (i) p and q have no common zeros in J (else W would have a zero row), and (ii) any zero of p and q in J is a simple zero (else W would have a zero column).

Moreover, if $W(x; p, q) \neq 0$ for all x in an open interval j, then (iii) any two consecutive zeros of p in J are separated by a zero of P, and (iv) any two consecutive zeros of q in J are separated by a zero of p. Indeed, if u and v are consecutive zeros of p then (as they are simple zeros) $p'(u)p'(v) < 0$. However, as W does not change sign in j we have $q(u)p'(u)q(v)p'(v) = W(u; p, q)W(v; p, q) > 0$, so that $q(u)q(v) < 0$. Thus q has a zero in (u, v) and (iii) holds. Of course, (iv) holds as we can interchange p and q. It is known that for orthogonal polynomials p_n, we have

$$W(x; p_k, p_{k+1}) > 0 \tag{4.1}$$

throughout $\mathbb{R}$ ([10], p. 43), and this implies that the zeros of p_n and p_{n+1} are interlaced.

5. Linear combinations of two adjacent polynomials

In this section, we discuss linear combinations of the form $ap_n + bp_{n+1}$. A comparison of the zeros of p_{n+1} and $ap_n + bp_{n+1}$ occurs in [2], and in [9], but not between two such linear combinations.

Theorem 5.1. *If* a *and* b *are real and not both zero, then every zero of* $ap_n + bp_{n+1}$ *is real and simple. Further,* $ap_n + bp_{n+1}$ *and* $cp_n + dp_{n+1}$ *have no common zeros, and their zeros are interlaced, unless one combination is a scalar multiple of the other.*

Proof. Theorem 3.1 shows that if a and b are real and not both zero then $ap_n + bp_{n+1}$ has at least n real zeros. Thus all of its zeros are real. Now suppose that $ad - bc \neq 0$. Then, from (4.1) and the linearity of the Wronskian,

$$W(x; ap_n + bp_{n+1}, cp_n + dp_{n+1}) = (ad - bc)W(x; p_n, p_{n+1}) \neq 0$$

on $\mathbb{R}$. Thus if $ad - bc \neq 0$ then $ap_n + bp_{n+1}$ and $cp_n + dp_{n+1}$ have only simple zeros, no common zeros, and their zeros are interlaced. □

We now consider the interlacing of the zeros of the combinations $ap_n + bp_{n+1}$ and $a_s p_s + \cdots + a_{n+1} p_{n+1}$. If we take $\omega = p_n + ap_{n+1}$ in the discussion above, then all Cotes numbers are positive ([7], Theorem III, p. 465). Thus we obtain the following corollary of Theorem 3.1.

Theorem 5.2. *Let* $P = a_s p_s + \cdots + a_{n+1} p_{n+1}$, *where* $a_s a_{n+1} \neq 0$, $1 \leqslant s \leqslant n+1$. *Let* $x_1, \ldots, x_{n+1}$ *be the zeros of* $p_n + ap_{n+1}$, *labelled so that* $x_j < x_{j+1}$. *Then there are at least* s *disjoint intervals* (x_j, x_{j+1}) *each of which contains a zero of* P.

We remark that Theorem 5.1 does not generalize to linear combinations of three polynomials. Consider, for example, the orthogonal polynomials generated by the recurrence relation $p_0(x) = 1$, $p_1(x) = x$ and $p_{n+2}(x) = xp_{n+1}(x) - p_n(x)$. Then $p_3(x)$, $p_4(x)$ and $p_5(x)$ are, respectively, $x^3 - 2x$, $x^4 - 3x^2 + 1$ and $x^5 - 4x^3 + 3x$. In this case the zeros of the two combinations $p_3 + p_4 + p_5$ and $7p_3 + 7p_4 + 5p_5$ are not interlaced.

6. Three polynomials of consecutive degrees

In [10] Shohat discussed the linear combination

$$P = ap_{n-2} + bp_{n-1} + p_n \tag{6.1}$$

and showed (Theorem VII, p. 472) that if $a < 0$ then P has only real simple zeros. This combination is also discussed in Theorem 2.5, [2], where it is shown that if $a < 0$ then the $n - 1$ zeros of p_{n-1} interlace with the zeros of P. Actually, both of these facts are easy consequences of the linearity of the Wronskian for

$$W(x; p_{n-1}, P) = W(x; p_{n-1}, p_n) - aW(x; p_{n-2}, p_{n-1}).$$

If $a < 0$ then $W(x; p_{n-1}, P) > 0$ throughout $\mathbb{R}$ and hence, by (i)–(iii) in Section 4, P has only simple zeros in $\mathbb{R}$, none of which are zeros of p_{n-1}, and the zeros of P and p_{n-1} are interlaced.

In Theorem VII (p. 472) Shohat asserts that if the polynomial ω in a quadrature formula is taken to be P as in (6.1), and if $a < 0$, then all of the Cotes numbers λ_j are positive. It follows that Theorem 3.1 is applicable, and since $s \leqslant n - 2$ and $d(\omega, \mu) = 2n - 3$, it implies that

$$N = \min\{s, (2n-3) + 1 - m\} \geqslant \min\{s, (2n-3) + 1 - n\} = s,$$

which yields the following result.

Theorem 6.1. *Let μ be an admissible Borel measure on $\mathbb{R}$, and let $\{p_n\}$ be the sequence of monic μ-orthogonal polynomials. Let $\omega = ap_{n-2} + bp_{n-1} + p_n$, where $a < 0$, with zeros $c_1, \ldots, c_n$, where $c_1 < \cdots < c_n$, and let $P = a_s p_s + \cdots + a_m p_m$, where $a_s a_m \neq 0$, $1 \leqslant s \leqslant m \leqslant n$ and $s \leqslant n - 2$. Then either P is a non-zero scalar multiple of ω, or there are s disjoint intervals (c_j, c_{j+1}) that contain at least one zero of P.*

We conclude this section by considering the extent to which the interlacing of the zeros of three arbitrary polynomials q_n, q_{n+1} and q_{n+2}, where each q_k has degree k, determines whether or not they can be embedded in an orthogonal sequence. It is known [12] that a necessary and sufficient condition for real polynomials p and q, of degrees n and $n+1$, respectively, to be embedded in an orthogonal sequence (with respect to some measure μ) is that their zeros interlace. It is easy to see that this result cannot be extended to three polynomials of degrees n, $n+1$ and $n+2$, so we now ask what the conditions are on three such polynomials that will guarantee that they lie in an orthogonal sequence? The answer to this lies in the following slightly more general result.

Theorem 6.2. *Let q_{n-1}, q_n and q_{n+1} be monic complex polynomials of degrees $n-1$, n and $n+1$, respectively. Suppose that q_n has n distinct zeros $y_1, \ldots, y_n$, and that these are not zeros of q_{n-1}. Then q_{n-1}, q_n and q_{n+1} satisfy a relation of the form*

$$q_{n+1}(z) = (z + a_n)q_n(z) + b_n q_{n-1}(z) \tag{6.2}$$

if and only if

$$\frac{q_{n+1}(y_1)}{q_{n-1}(y_1)} = \cdots = \frac{q_{n+1}(y_n)}{q_{n-1}(y_n)}. \tag{6.3}$$

The proof is easy. Suppose throughout that the q_k satisfy the assumptions in Theorem 6.2. If they also satisfy (6.2) then, with $z = y_k$, we obtain (6.3). Now suppose that the q_k satisfy (6.3), and let b_n be the common value of the ratios in (6.3). Then $q_{n+1}(z) - b_n q_{n-1}(z)$ is a monic polynomial of degree $n+1$ that is zero at each y_j, and so is divisible by q_n. Thus (6.2) holds for some a_n. □

7. Remarks on Shohat's result

We consider Shohat's result that any linear combination of μ-orthogonal polynomials $P = a_s p_s + \cdots + a_n p_n$, where $s \leqslant n$ and $a_s a_n \neq 0$, has at least s distinct zeros in $I(\mu)$, and also the fact that this is best possible. To see that P has at least s distinct zeros in $I(\mu)$, let $p^*(x) = (x - y_1)\cdots(x - y_v)$, where $y_1, \ldots, y_v$ are the distinct odd-order zeros of P that lie in $I(\mu)$ (and $p^*(x) \equiv 1$ if no such y_i exist). Then

$$a_n \int_{\mathbb{R}} P(x)p^*(x)\, d\mu(x) = \int_{\mathbb{R}} \left[\frac{a_n P(x)}{p^*(x)}\right] p^*(x)^2\, d\mu(x) > 0$$

because the two factors in the integrand of the second integral are non-negative in $I(\mu)$. This implies that P is not μ-orthogonal to p^*, so that p^* must have degree at least s. Thus *P has at least s distinct odd-order zeros in $I(\mu)$*.

We give a new, and simple, proof that this result is best possible. First, we recall Descartes' Theorem that *the number of positive zeros of a real polynomial is at most the number of sign changes in the sequence of its coefficients*. Here a *sign change* means passing from a positive coefficient to a negative coefficient, or from a negative coefficient to a positive coefficient, when, if they are not adjacent coefficients, all coefficients between them are zero. Let us now apply this to the sequence L_n^α of Laguerre polynomials. These are defined by

$$\begin{aligned} n!L_n^\alpha(x) = &\left[\binom{n}{0}(\alpha+1)\cdots(\alpha+n)\right] + \left[(-x)\binom{n}{1}(\alpha+2)\cdots(\alpha+n)\right] \\ &+ \left[(-x)^2\binom{n}{2}(\alpha+3)\cdots(\alpha+n)\right] + \cdots \\ &+ \left[(-x)^{n-1}\binom{n}{n-1}(\alpha+n)\right] + (-x)^n \end{aligned}$$

and they form an orthogonal sequence with respect to the measure $d\mu(x) = x^\alpha e^{-x}dx$ on $\mathbb{R}$ when $\alpha > -1$ but not otherwise. If $\alpha > -1$ then all of the products $(\alpha+k)\cdots(\alpha+n)$ are positive, and there are n sign changes in $L_n(x,\alpha)$; thus (as we already know) $L_n(x,\alpha)$ has n zeros in $(0,+\infty)$. As α decreases, the number of sign changes of L_n^α decreases by one as α passes through each negative integer $-1,\dots,-n$. Thus if $N \in \{1,\dots,n\}$, and $-N-1 < \alpha < -N$, then $L_n^\alpha(x)$ has at most $n-N$ zeros in $(0,+\infty)$. However, the Laguerre polynomials satisfy the functional relation

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) - L_{n-1}^{\alpha+1}(x),$$

(see [1] and [2]) and it is evident from this that if $-N-1 < \alpha < -N$ then L_n^α is a linear combination of $L_j^{\alpha+N}$, where $j = n-N,\dots,n$. We now see that this linear combination has (by Shohat's argument) at least $n-N$ zeros in $(0,+\infty)$, and (by Descartes' Theorem) at most $n-N$ zeros there. In conclusion, if $-(N+1) < \alpha < -N$ and $1 \leqslant N \leqslant n$, then L_n^α is a linear combination of $L_{n-N}^{\alpha+N},\dots,L_n^{\alpha+N}$ and this linear combination has exactly $n-N$ zeros in $(0,+\infty)$.

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